High School Mathematics Contest

Elon University Mathematics Department

Saturday, March 23, 2013

1. Find the reflection (or mirror image) of the point $(-3,0)$ about the line $y = 3x - 1$.

   (a) $(3, 0)$.
   (b) $(3, -3)$.
   (c) $(3,-\sqrt{2})$.
   (d) $(3, -1)$.
   (e) $(3, -2)$.

2. Find the sum of the coefficients of $(a + b + c + d)^{10}$

   (a) 144.
   (b) 1024.
   (c) 65,536.
   (d) 262,144.
   (e) 1,048,576.

3. The following statements were made on the same day:

   - It was Monday yesterday
   - Today is Thursday
   - The day after tomorrow will be Friday
   - Tomorrow will be Saturday
   - The day before yesterday was Tuesday

   Given that the number of statements above that are correct uniquely determines the day of the week on which the statements were made, on what day of the week were the statements made?

   (a) Monday.
   (b) Tuesday.
   (c) Wednesday.
   (d) Thursday.
   (e) None of the previous choices.

4. Assume that b and c are integers greater than one. In base b, c^2 is written as 10. Then b^2, when written in base c, is

   (a) 100.
   (b) 101.
   (c) 1010.
   (d) 10000.
   (e) cannot be determined.

5. Let z be a complex number and $\overline{z}$ be its complex conjugate. If $\overline{z}/z = i$, how many solutions are there for z?

   (a) 0.
   (b) 1.
   (c) 2.
   (d) 4.
   (e) Infinite number.

6. Let $0 \leq a \leq 3$. Let b be the number in base 4 represented by $a32$. Let c be the number in base 16 represented by 237. What is the value of the second to last digit in $b + c$ in base 16?

   (a) $a + 4$.
   (b) $a + 3$.
   (c) 6.
   (d) a.
   (e) None of the above.

7. Two dogs, each traveling 15 ft/sec, run toward each other from 1200 feet apart. As they run, a bee flies from the nose of one dog to the nose of the other at 25 ft/sec. The bee flies between the dogs in this manner until it is crushed when the dogs collide. How far did the bee fly?

   (a) 600 ft.
   (b) 800 ft.
   (c) 1000 ft.
   (d) 1200 ft.
   (e) none of the above.

8. Let $a, b, c \in \mathbb{R}$ such that they satisfy the three equations below:

$$a + \tfrac{1}{bc} = \tfrac{1}{5} \quad b + \tfrac{1}{ac} = -\tfrac{1}{15} \quad c + \tfrac{1}{ab} = \tfrac{1}{3}$$

   What is the value of the quotient $\dfrac{c-b}{c-a}$?

   (a) -5.
   (b) -3.
   (c) 1.
   (d) 3.
   (e) 5.

9. Suppose 6 chess players are paired up to play matches and in each of the 3 matches, there is a winner. How many ways can the 6 players be ranked from first to last so that the ranking does not contradict any of the matches. For example if Dani beat Poh in a match, Dani has to be above Poh in every ranking.

   (a) 1.
   (b) 36.
   (c) 90.
   (d) 120.
   (e) 720.

10. The asymptotes to a hyperbola have the following equations:

$$y = 3x - 7$$
$$y = -3x + 5$$

Find the possible values for the eccentricity of the hyperbola.

(a) 1.1 and 3.2.
(b) 1.2 and 3.3.
(c) 1.3 and 3.1.
(d) 2.1 and 4.5.
(e) 3.1 and 3.6.

11. If 9 red balloons, 6 blue balloons and 12 yellow balloons are to be distributed between 4 children, how many distributions are possible if every child must receive at least one balloon of each color?

(a) 27.
(b) 648.
(c) 92,400.
(d) 369,600.
(e) 935,550.

12. An equilateral triangle is inscribed in a circle. What percentage of the area of the circle is covered by the triangle? Round to the nearest whole percent.

(a) 59%.
(b) 53%.
(c) 44%.
(d) 41%.
(e) 38%.

13. Find the sum $\sqrt[3]{5 + 2\sqrt{13}} + \sqrt[3]{5 - 2\sqrt{13}}$.

(a) $\sqrt{13} - 1$.
(b) 1.
(c) $2\sqrt{13} + 5$.
(d) 2.
(e) none of the above.

14. The only value of a for which the simultaneous equations

$$\begin{aligned} 2x + 3y &= 5 \\ x + ay &= 2 \end{aligned}$$

would have no solution is

(a) 2.
(b) 0.
(c) $\frac{3}{2}$.
(d) $\frac{1}{2}$.
(e) $-\frac{1}{2}$.

15. Nautical flags are specially designed flags made up of several colors which can be used to signal from ship to ship, or ship to shore. Suppose there are 4 red, 5 blue and 8 yellow flags. How many different arrangements can be made if ONLY 16 of the 17 flags can be used on a vertical flag pole?

(a) 180,180.
(b) 3,063,060.
(c) 20,922,789,888,000.
(d) 355,687,428,096,000.
(e) Not enough information given.

16. Let A, B, C, D, and E be integers such that $A^4 + B^4 + C^4 + D^4 + E^4 < 20$ and $ABCDE$ is odd. What is the largest number of factors $x - y$ (where x and y take values from A, B, C, D, and E) that can be multiplied with a non-zero product?

(a) 3.
(b) 5.
(c) 6.
(d) 10.
(e) 24.

17. What is the remainder when 2013^{2013} is divided by 10?

(a) 0.
(b) 1.
(c) 3.
(d) 7.
(e) 9.

18. How many solutions does the equation

$$\sin(x)\sin(2x)\sin(3x)\cdots\sin(11x)\sin(12x) = 0$$

have in the interval $(0,\pi]$?

(a) 11.
(b) 12.
(c) 24.
(d) 46.
(e) 68.

19. How many lines are tangent to both of the following curves?

$$x^2 + y^2 - 2y = 0$$
$$x^2 + y^2 - 10y + 16 = 0$$

(a) 0.
(b) 1.
(c) 2.
(d) 3.
(e) 4.

20. I have two dice, one red and one blue. When the two dice are rolled, the probability that the number showing on the red die is larger than the number showing on the blue die is

(a) $\frac{5}{12}$.
(b) $\frac{4}{9}$.
(c) $\frac{1}{2}$.
(d) $\frac{19}{36}$.
(e) $\frac{2}{3}$.

21. A box contains only white and black balls. Let p be the probability that a ball selected at random is black. Each time a ball is selected, it is placed back in the box before selecting the next ball. What is the probability that two of the four balls selected are black and two are white?

(a) $6p^2(1-p)^2$.
(b) $4p^2(1-p)^2$.
(c) $p^2(1-p)^2$.
(d) $4p^2(1-p^2)$.
(e) $\frac{1}{2}$.

22. What is the probability that a set of 2013 integers selected randomly without replacement from the set of all positive integers contains two distinct numbers whose difference is divisible by 729?

(a) 0.
(b) $\frac{1}{2012}$.
(c) $\frac{1}{2013}$.
(d) 1.
(e) Not enough information is given.

23. Find m so that the area of the triangle bounded by $y = 0$, $x = 10$, and $y = mx$ is 3.

(a) 0.06.
(b) 0.08.
(c) 0.09.
(d) 0.10.
(e) 0.12.

24. If $R_n = \frac{1}{2}(a^n + b^n)$, where $a = 3 + 2\sqrt{2}$, $b = 3 - 2\sqrt{2}$, and $n = 0, 1, 2, \ldots$, then find the units digit of R_{12345}.

    (a) 1.
    (b) 3.
    (c) 6.
    (d) 7.
    (e) 9.

25. Let x and y be real numbers. If $xy = 10$ and $x^2 + y^2 = 1$, then $\frac{y}{x} + \frac{x}{y} =$?

    (a) 0.
    (b) $\frac{1}{10}$.
    (c) $-\frac{1}{10}$.
    (d) 10.
    (e) Does not exist.

26. What is the measure of the acute angle between the hour and minute hands of a correctly working clock at 4:18?

    (a) $12°$.
    (b) $15°$.
    (c) $18°$.
    (d) $21°$.
    (e) $24°$.

27. Let $f(x) = \ln(e^x + 1)$. Let $g_1(x) = f(x)$ and $g_{n+1}(x) = f(g_n(x))$ for $n \geq 1$. How many real solutions are there to $g_{10}(x) = x$?

    (a) 0.
    (b) 1.
    (c) 2.
    (d) 3.
    (e) 6.

28. An odd integer between 600 and 800 is divisible by 7 and is also divisible by 9. What is the sum of its digits?

    (a) 7.
    (b) 12.
    (c) 18.
    (d) 21.
    (e) 27.

29. A certain function f satisfies $f(x) + 2f(6 - x) = x$ for all real numbers x. The value of $f(1)$ is

    (a) -9.
    (b) 1.
    (c) 2.
    (d) 3.
    (e) not possible to determine.

30. In quadrilateral $ABCD$ with diagonals AC and BD intersecting at O, $BO = 4$, $OD = 6$, $AO = 8$, $OC = 3$, and $AB = 6$. Find AD.

    (a) 10.
    (b) $\frac{\sqrt{98}+8}{3}$.
    (c) $\frac{45}{2}$.
    (d) $\sqrt{166}$.
    (e) $\sqrt{\frac{45}{2}}$.

31. Ten writers covering a basketball league vote for the Most Valuable Player of the league by listing their top three choices. A first place vote earns five points, a second place vote earns three points, and a third place vote earns one point. Julie Jubilee received 37 points. How many writers listed her on their ballots?

    (a) 7.
    (b) 8.
    (c) 9.
    (d) 10.
    (e) There is not enough information to determine this..

32. What is the remainder of $2^{10,000}$ when divided by 21?

    (a) 1.
    (b) 3.
    (c) 13.
    (d) 16.
    (e) 20.

33. Given that $P(x)$ is a polynomial such that $P(x^2+1) = x^4 + 5x^2 + 3$, what is $P(x^2-1)$?

    (a) $x^4 + x^2 - 3$.
    (b) $x^4 + 5x^2 - 1$.
    (c) $x^2(x+1)(x-1)$.
    (d) $x^4 + x^2 + 1$.
    (e) $x^4 - x^2 - 1$.

34. Consider taking three distinct planar slices of a torus (perfect bagal/donut). Each slice must individual divide the torus into at least two pieces. What is the maximum number of pieces the torus can be divided into?

    (a) 6 pieces.
    (b) 8 pieces.
    (c) 9 pieces.
    (d) 10 pieces.
    (e) 11 or more pieces.

35. Let $ABCD$ be a square. Point P is on the side $\overline{AB}$ and $AP = 2BP$. Point Q is on the side $\overline{BC}$ with $BQ = 2CQ$. What is the sum of the measures of the angles $\angle QAB$, $\angle PDQ$, and $\angle PCB$?

    (a) $60°$.
    (b) $75°$.
    (c) $100°$.
    (d) $120°$.
    (e) none of the previous choices.