

2007 High School Math Contest

Elon University

March 31, 2007

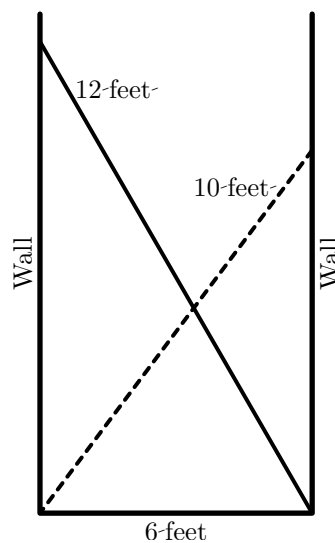
Note : In general, figures are drawn not to scale! All decimal answers are rounded to the nearest hundredth, and all triangles can be taken to be non-degenerate.

1. In a rectangle we decrease its length by 2 and increase its width by 6 to form a square of area 289. Find the perimeter of the rectangle.

- (a) 60.
- (b) 64.
- (c) 68.
- (d) 72.
- (e) 76.

2. There is a 6 foot wide alleyway. Both of the walls in the alley are perpendicular to the ground. Two ladders, one 10 feet long and the other 12 feet, are propped up from the opposite corners to the adjacent wall, forming an X-shape. All four feet of each ladder are firmly touching the corner or the wall. The two ladders are also touching each other at the intersection of the X-shape. What is the distance from the point of intersection to the ground?

- (a) 3.39 feet.
- (b) 4.52 feet.
- (c) 5.00 feet.
- (d) 7.38 feet.
- (e) 8.29 feet.



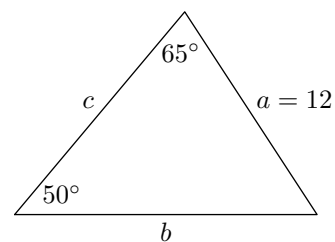
3. Which of the following statements are not true equations for all real numbers x and y ?

- I. $\ln(xy) = \ln(x) + \ln(y)$
- II. $e^{x+y} = e^x e^y$
- III. $\tan^2(x) + 1 = \sec^2(x)$

- (a) I.
- (b) II.
- (c) III.
- (d) More than one.
- (e) None of them.

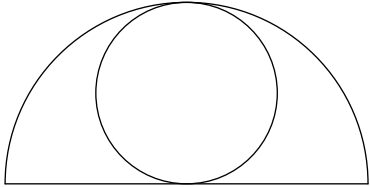
4. Find the length of the side b in the figure.

- (a) 12.00.
- (b) 12.14.
- (c) 13.24.
- (d) 13.75.
- (e) 14.20.



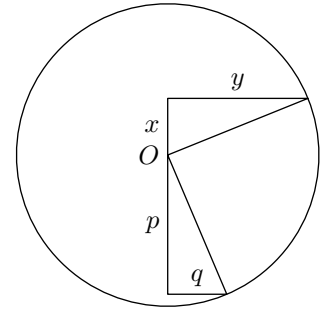
5. If (x, y) is a solution to the system of equations $xy = 6$ and $x^2y + xy^2 + x + y = 63$, then the point (x, y) must lie on a circle centered at the origin. What is the radius of this circle?
- (a) 2.45.
 - (b) 6.00.
 - (c) 7.21.
 - (d) 7.94.
 - (e) 8.31.
6. At the end of 2006, Betty was half as old as her father. The sum of the years in which they were born is 3946. How old will Betty be at the end of 2020?
- (a) 36.
 - (b) 44.
 - (c) 58.
 - (d) 72.
 - (e) 80.
7. What is the units digit of the integer representation of 3^{2007} ?
- (a) 1.
 - (b) 3.
 - (c) 6.
 - (d) 7.
 - (e) 9.
8. Two six-sided dice each have two red, two yellow, and two blue faces. If we roll the dice, what is the probability that both dice show matching colors?
- (a) $\frac{1}{72}$.
 - (b) $\frac{1}{36}$.
 - (c) $\frac{1}{12}$.
 - (d) $\frac{1}{3}$.
 - (e) $\frac{1}{2}$.

9. The consecutive angles of a trapezoid form an arithmetic sequence. If the largest angle is 105° , then the smallest angle is
- (a) 65° .
 - (b) 70° .
 - (c) 75° .
 - (d) 80° .
 - (e) 85° .
10. A ball is painted entirely one color, and is placed in a sealed box. Exactly three of the following statements are true, and one is false.
- I. The ball is red.
 - II. The ball is not yellow.
 - III. The ball is green.
 - IV. The ball is not purple.
- Which one of the following must be correct?
- (a) I is true.
 - (b) II is false.
 - (c) III is false.
 - (d) IV is true.
 - (e) I and III are both false.
11. A cylindrical can just holds three tennis balls stacked on top of each other (so they touch the sides, top and bottom of the can). The height of the stacked balls is 9 inches. What percent of the can is empty space outside of the balls?
- (a) $4\frac{1}{2}\%$.
 - (b) 15%.
 - (c) 27%.
 - (d) 31%.
 - (e) $33\frac{1}{3}\%$.
12. A certain bacteria is known to triple every 3 hours. How long does it take to quadruple? (Round to the nearest minute.)
- (a) 0h 52m.
 - (b) 2h 23m.
 - (c) 3h 47m.
 - (d) 4h 0m.
 - (e) 4h 31m.
13. First x is chosen at random from the set $\{1, 2, 3, \dots, 99, 100\}$, and then y is chosen at random from the same full set. The probability that the integer $3x + 7y$ has units digit 8 is
- (a) $\frac{1}{16}$.
 - (b) $\frac{1}{8}$.
 - (c) $\frac{3}{16}$.
 - (d) $\frac{1}{5}$.
 - (e) $\frac{1}{4}$.

14. For how many integers N between 1 and 1990 (inclusive) is the improper fraction $\frac{N^2 + 7}{N + 4}$ *not* in lowest terms?
- (a) 0.
(b) 86.
(c) 90.
(d) 104.
(e) 105.
15. A small circle just fits inside a semicircle. What is the ratio of the area of the small circle to the area of the semicircle containing it?
- (a) 1:1.
(b) 1:2.
(c) 1:3.
(d) 2:3.
(e) 3:4.
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16. Jason has taken 4 tests in his French class. If he gets a 70 on the 5th test, his test average will be 74. What is his current test average?
- (a) 70.
(b) 74.
(c) 75.
(d) 77.
(e) 80.
17. How many five-digit whole numbers that are multiples of 5 can be formed from the digits 1, 2, 3, 4, and 5 using each of the digits exactly once?
- (a) 6.
(b) 24.
(c) 30.
(d) 60.
(e) 120.
18. If $|a - 3| = b$, $0 < a < 3$ and $a \neq b$, then $a - b =$
- (a) 3.
(b) -3 .
(c) $3 - 2b$.
(d) $2b - 3$.
(e) $|2b - 3|$.

19. The sides of one triangle have lengths 8, 10, and m . The sides of another triangle have lengths 8, 10, and n . What is the smallest positive number that is not a possible value of $|m - n|$?
- (a) 2.
 - (b) 8.
 - (c) 10.
 - (d) 16.
 - (e) 18.
20. The surface of a cube with edges of length 6 inches is painted. It is then cut into 1 in^3 small cubes. How many of these smaller cubes have exactly one painted face?
- (a) 35.
 - (b) 48.
 - (c) 96.
 - (d) 150.
 - (e) 216.
21. How many positive integers, when expressed in base seven, are equal to the sum of their own digits?
- (a) 4.
 - (b) 5.
 - (c) 6.
 - (d) 7.
 - (e) 8.
22. A triangle with positive area and integral sides has perimeter 8. The area of the triangle is
- (a) $2\sqrt{2}$.
 - (b) $\frac{16}{9}\sqrt{3}$.
 - (c) $2\sqrt{3}$.
 - (d) 4.
 - (e) $4\sqrt{2}$.
23. A cubic polynomial $p(x)$ with leading coefficient 1 has three real roots. The median of the roots is 0, as is the average. If the range of the set of roots is 10, what is the coefficient of x in $p(x)$?
- (a) -25 .
 - (b) -10 .
 - (c) -5 .
 - (d) 0.
 - (e) can not be determined.

24. In the figure at right, the two triangles are right triangles with sides of length x , y , p , and q as shown. The shared point, O , is the center of the circle and the indicated vertices are on the circle. Given that $x^2 + y^2 + p^2 + q^2 = 72$, find the circumference of the circle.



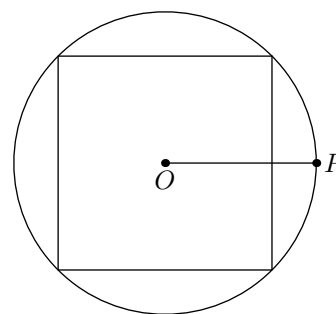
- (a) 8π .
- (b) 9π .
- (c) 12π .
- (d) 24π .
- (e) 36π .
25. Let $p(x)$ be a polynomial such that $p(\cos(x)) = \cos(3x)$ for all x . Which of the following is a factor for $p(x)$?
- (a) $x - \frac{\pi}{6}$.
- (b) $x - \frac{\pi}{3}$.
- (c) $x - \frac{\sqrt{2}}{2}$.
- (d) $x - \frac{\sqrt{3}}{2}$.
- (e) $x - \frac{\pi}{2}$.
26. Four whole numbers, when added three at a time give the sums 180, 197, 208, and 222. What is the largest of the four numbers?
- (a) 77.
- (b) 83.
- (c) 89.
- (d) 95.
- (e) 101.
27. Let a be a positive number. What values of a have a solution x for the equation $a^x = x$?
- (a) $(0, 0.50]$.
- (b) $(0, 0.71]$.
- (c) $(0, 1.00]$.
- (d) $(0, 1.41]$.
- (e) $(0, 1.44]$.
28. Let $f(x) = 1/(1-x)$. Define $f_1(x) = f(x)$ and $f_{n+1}(x) = f(f_n(x))$ for $n \geq 1$. What is $f_{2007}(x)$?
- (a) x .
- (b) $\frac{1}{1-x}$.
- (c) $\frac{x-1}{x}$.
- (d) $\frac{x}{x-1}$.
- (e) $\frac{x}{1-x}$.

29. Suppose you have a large bar of chocolate, scored into 200 individual squares to make up a rectangular slab 10 pieces wide and 20 pieces long. You start to break it into pieces. You can make the breaks in any order you like. What is the least number of breaks you need to make to divide the slab into 200 individual square pieces?

(a) 8.
(b) 50.
(c) 100.
(d) 199.
(e) 200.

30. Consider the circle in the figure with center O . If $|OP| = \sqrt{2}$, what is the perimeter of the inscribed square?

(a) $\frac{\sqrt{2}}{4}$.
(b) $2\sqrt{2}$.
(c) $4\sqrt{2}$.
(d) 8.
(e) $2\sqrt{2}\pi$.



31. Suppose you have stamps with values 5 cents and 8 cents. What is the minimum integer x so that for any integer $y \geq x$ there is a combination of stamps whose values add to y ?

(a) 21.
(b) 23.
(c) 28.
(d) 32.
(e) 40.

32. Points A and B are 5 units apart. How many lines that lie in a given plane containing A and B are 2 units from A and 3 units from B ?

(a) 0.
(b) 1.
(c) 2.
(d) 3.
(e) more than 3.

33. A race car driver drives half the distance at 150 mph and the other half at 100 mph. What was the driver's average speed?

(a) 105.
(b) 110.
(c) 115.
(d) 120.
(e) 125.

34. Let \diamond be a binary operation on the set of positive integers, so that

$$a \diamond b = a^2 + b^2 - ab.$$

How many ordered pairs (x, y) are there that satisfy the equation $x \diamond y = x$?

- (a) 0.
 - (b) 1.
 - (c) 2.
 - (d) 3.
 - (e) infinity.
35. In the diagram, O is the center of the circle, $\angle OAB = 20^\circ$ and $\angle OCB = 70^\circ$. Then $\angle ABC =$

- (a) 50° .
- (b) 60° .
- (c) 70° .
- (d) 80° .
- (e) 90° .

