2009 High School Mathematics Contest

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1. For integers $a, b,$ and c, define $\&(a, b, c) = \frac{ab}{a+c}$. What is $\&(1, -2, -3)$
   (a) -1.
   (b) 0.
   (c) 1.
   (d) 2.
   (e) 3.

2. A student is taking Calculus and scores 62, 92, 71, and 83 on the first four tests. How many different scores (whole numbers less than or equal to 100) can the student score on the last test and end up with a median test score of 83?
   (a) 1.
   (b) 8.
   (c) 17.
   (d) 18.
   (e) none of the above.

3. Which of the following is equivalent to "If P is true then Q is false"?
   (a) "P is true or Q is false".
   (b) "If Q is false then P is true".
   (c) "If P is false then Q is true".
   (d) "If Q is true then P is false".
   (e) "If Q is true then P is true".

4. Suppose that a and b are two non-zero real numbers with $a < b$. Which of the following statements must be true regarding a and b?

   I. $a^2 < b^2$ II. $a < |b|$ III. $\frac{1}{b} < \frac{1}{a}$

   (a) I only.
   (b) II only.
   (c) III only.
   (d) I and II.
   (e) I and III.

5. The number 10! (10 is written in base 10), when written in the base 12 system, ends with exactly k zeros. The value of k is:
   (a) 1.
   (b) 2.
   (c) 3.
   (d) 4.
   (e) 5.

6. Three standard dice are rolled, and someone tells you that the total of the top faces is 13. What is the probability that any one die has the numbers 4 or 5 on the top face?
   (a) 15/21.
   (b) 17/21.
   (c) 18/21.
   (d) 19/21.
   (e) not enough information.

7. Let S be the statement

   If the sum of the digits of the whole number n is divisible by 6, then n is divisible by 6.

   A value of n which shows S to be false is
   (a) 30.
   (b) 33.
   (c) 40.
   (d) 42.
   (e) none of these.

8. A group of people was interviewed after the Olympics to see what Olympic sports they had watched on TV. The following information was revealed.
   (i) 41% watched Running
   (ii) 34% watched Swimming
   (iii) 27% watched Gymnastics
   (iv) 19% watched Running and Swimming
   (v) 17% watched Running and Gymnastics
   (vi) 12% watched Swimming and Gymnastics
   (vii) 6% watched Running and Swimming and Gymnastics

   Calculate the percentage of people in the group that watched none of the 3 Olympic sports mentioned.
   (a) 6%.
   (b) 34%.
   (c) 40%.
   (d) 94%.
   (e) none of these.

9. If $f(2x) = \frac{2}{2+x}$ for all $x > 0$, then $2f(x) =$

   (a) $\frac{2}{1+x}$.
   (b) $\frac{2}{2+x}$.
   (c) $\frac{4}{1+x}$.
   (d) $\frac{4}{2+x}$.
   (e) $\frac{8}{4+x}$.

10. There are two cards; one is red on both sides and the other is red on one side and blue on the other. The cards have the same probability (1/2) of being chosen, and one is chosen and placed on the table. If the upper side of the card on the table is red, then the probability that the under-side is also red is

    (a) 1/4.
    (b) 1/3.
    (c) 1/2.
    (d) 2/3.
    (e) 3/4.

11. Which of the following triangles cannot exist?

    (a) An acute right triangle.
    (b) An isosceles right triangle.
    (c) An obtuse right triangle.
    (d) A scalene right triangle.
    (e) A scalene obtuse triangle.

12. In an odd form of a duel, the two contestants start back to back, but pace away from each other at a $100°$ angle. They take ten paces, turn and fire mathematics problems at each other. In one duel, one contestant has a pace 0.6 meters long and the other's is 0.8 meters long. When the contestant with the longer stride turns and aims at the other, what is the angle that his sighting makes with his path (to the nearest tenth of a degree)?

    (a) $10°$.
    (b) $30.5°$.
    (c) $33.2°$.
    (d) $36.7°$.
    (e) $45°$.

13. If $\log_9(x) = 2\log_3(x)$, then what is $x^3 - 6x^2 + 11x - 6$?

    (a) -6.
    (b) -1.
    (c) 0.
    (d) 6.
    (e) Not enough information.

14. If the median of three numbers is 5, the median of their squares is 49, and the mean of their magnitudes is 7, what is the mean of their squares, rounded to the nearest integer?

    (a) 35.
    (b) 49.
    (c) 50.
    (d) 52.
    (e) Not enough information.

15. Let $f(x) = ax^7 + bx^3 + cx - 5$, where a, b, and c are constants. If $f(-7) = 7$, then $f(7)$ equals

    (a) -17.
    (b) -7.
    (c) 14.
    (d) 21.
    (e) not uniquely determined.

16. Consider the function f defined by $f(x) = 2x + 3$. Suppose that a sequence $\{x_n\}$ is defined by $x_0 = 3$ and $x_n = f(x_{n-1}) \pmod 5$. The value of x_{101} is:

    (a) 0.
    (b) 1.
    (c) 2.
    (d) 3.
    (e) 4.

17. If the expression $\begin{vmatrix} a & c \\ d & b \end{vmatrix}$ has the value $ab - cd$ for all values of a, b, c, and d, then the equation $\begin{vmatrix} 2x & 1 \\ x & x \end{vmatrix} = 3$:

    (a) is satisfied for no values of x.
    (b) is satisfied for only one value of x.
    (c) is satisfied for two values of x.
    (d) is satisfied for an infinite number of values of x.
    (e) none of these.

18. A positive integer N with three digits in its base ten representation is chosen at random, with each three digit number having an equal chance of being chosen. The probability that $\log_2(N)$ is an integer is

    (a) 0.
    (b) 1/450.
    (c) 1/300.
    (d) 3/899.
    (e) 1/225.

19. The radius of a cylindrical box is 8 inches and the height is 3 inches. The number of inches that may be added to either the radius or the height to give the same non-zero increase in volume is:
    (a) 1.
    (b) $5\frac{1}{3}$.
    (c) any number.
    (d) non-existent.
    (e) none of these.

20. An engineer said she could finish a highway section in 3 days with her present supply of a certain type of machine. However, with 3 more of these machines, the job could be done in 2 days. If the machines all work at the same rate, how many days would it take to do the job with one machine?
    (a) 6.
    (b) 12.
    (c) 15.
    (d) 18.
    (e) 36.

21. The three-digit number $2a3$ is added to the number 326 to give the three-digit number $5b9$. If $5b9$ is divisible by 9, then $a + b$ equals:
    (a) 2.
    (b) 4.
    (c) 6.
    (d) 8.
    (e) 9.

22. A closet contains 8 pairs of shoes (therefore a total of 16 shoes). Assume that 6 shoes are chosen at random without replacement. What is the probability that there will be no matching pairs (rounded to three decimal places)?
    (a) 0.003.
    (b) 0.007.
    (c) 0.028.
    (d) 0.224.
    (e) none of the above.

23. How many subsquares can you form on chessboard with $n \times n$ unit squares? Subsquares must be formed from full unit squares.
    (a) $\dfrac{n(n-1)}{4}$.
    (b) $\dfrac{n(n+1)}{6}$.
    (c) $\dfrac{n(2n-1)}{9}$.
    (d) $\dfrac{n(n+1)(2n+1)}{6}$.
    (e) $\dfrac{n(2n+1)}{9}$.

24. If Sawyer and Kate can paint a very large room in 7 hours and 5 hours, respectively, how fast can they paint the room together?

    (a) 145 minutes.
    (b) 160 minutes.
    (c) 175 minutes.
    (d) 185 minutes.
    (e) 200 minutes.

25. Suppose Jane is twice as old as Jackie today. In 5 years, their combined age will be 4 times the age Jackie was 5 years ago. How old is Jackie today?

    (a) 27.
    (b) 28.
    (c) 29.
    (d) 30.
    (e) 31.

26. Let a, b, c, d be integers with $a < 2b, b < 3c,$ and $c < 4d$. If $d < 100$, then the largest possible value for a is

    (a) 2367.
    (b) 2375.
    (c) 2391.
    (d) 2399.
    (e) 2400.

27. Joe's happiness is proportional to the function w^2c, where w stands for daily consumption of glasses of wine, and c stands for daily consumption of number of cigars. Wine costs \$3 a glass and cigars cost \$2 each. His daily happiness budget is \$100. How should he divide his money between wine and cigars to maximize his happiness? Assume he does not have to buy integer numbers of either cigars or wine.

    (a) He should spend \$10.99 on wine and \$89.01 on cigars..
    (b) He should spend \$27.32 on wine and \$72.68 on cigars..
    (c) He should spend \$40.49 on wine and \$59.51 on cigars..
    (d) He should spend \$66.67 on wine and \$33.33 on cigars..
    (e) He should spend \$50.00 on wine and \$50.00 on cigars..

28. A circular piece of metal of maximum size is cut out of a square piece and then a square piece of maximum size is cut out of the circular piece. If the final square is all that is used, the total amount of metal wasted is:

    (a) $\frac{1}{4}$ the area of the circular piece.
    (b) $\frac{1}{2}$ the area of the circular piece.
    (c) $\frac{1}{4}$ the area of the original square.
    (d) $\frac{1}{2}$ the area of the original square.
    (e) none of these.

29. What is the volume (to the nearest hundredth) of an tetrahedron with edge length of 1?

   (a) 0.12 units cubed.
   (b) 0.28 units cubed.
   (c) 2.85 units cubed.
   (d) 3.76 units cubed.
   (e) 7.65 units cubed.

30. Consider the 6 complex solutions of $z^6 - 64 = 0$ of the form $a + bi$. What is the product of those solutions with $a > 0$?

   (a) 1.
   (b) 4.
   (c) 8.
   (d) $\frac{1}{2} + \frac{\sqrt{3}}{2}i$.
   (e) $1 + i$.

31. An equivalent of the expression

$$\left(\frac{x^2+1}{x}\right)\left(\frac{y^2+1}{y}\right)+\left(\frac{x^2-1}{y}\right)\left(\frac{y^2-1}{x}\right), \qquad xy \neq 0,$$

   is:

   (a) 1.
   (b) $2xy$.
   (c) $2x^2y^2 + 2$.
   (d) $2xy + \dfrac{2}{xy}$.
   (e) $\dfrac{2x}{y} + \dfrac{2y}{x}$.

32. If r_1, r_2, and r_3 are the roots of the polynomial $x^3 + 5x^2 + 3x - 2$, find the average of their reciprocals.

   (a) 0.5.
   (b) 0.6.
   (c) 0.8.
   (d) 0.9.
   (e) 1.0.

33. If $\dfrac{xy}{x+y} = a$, $\dfrac{xz}{x+z} = b$, and $\dfrac{yz}{y+z} = c$, where a, b, and c are other than zero, then x equals:

(a) $\dfrac{abc}{ab+ac+bc}$.
(b) $\dfrac{2abc}{ab+bc+ac}$.
(c) $\dfrac{2abc}{ab+ac-bc}$.
(d) $\dfrac{2abc}{ab+bc-ac}$.
(e) $\dfrac{2abc}{ac+bc-ab}$.

34. A box contains 11 balls, numbered $1, 2, 3, \ldots, 11$. If 6 balls are drawn simultaneously at random, what is the probability that the sum of the numbers drawn is odd?

(a) $\frac{100}{231}$.
(b) $\frac{115}{231}$.
(c) $\frac{1}{2}$.
(d) $\frac{118}{231}$.
(e) $\frac{6}{11}$.

35. Given that a, b, and c are integers larger than 1, and that $a^{(b^c)} = \left(a^b\right)^c$, find $b + c$.

(a) 4.
(b) 5.
(c) 6.
(d) 7.
(e) 8.