

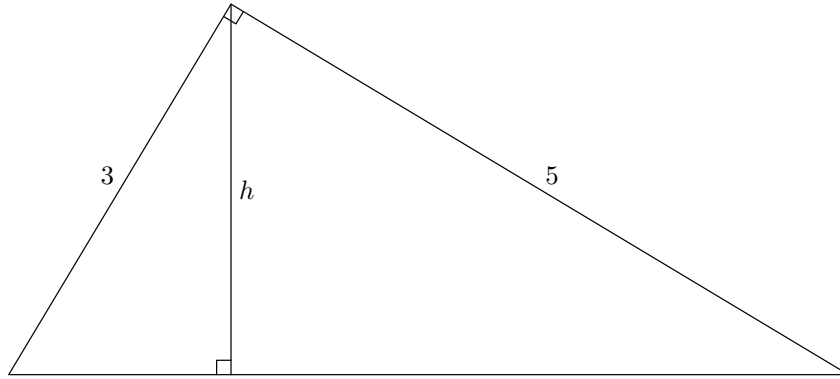
# High School Mathematics Contest

Elon University Mathematics Department

Saturday, March 13, 2010

1. John is as old as Eli and Walter combined right now. In 20 years, Eli will be twice as old as Walter. In 4 years, John will be five times as old as Walter. What is the sum of the ages of John, Eli and Walter right now?
  - (a) 2.
  - (b) 24.
  - (c) 26.
  - (d) 52.
  - (e) None of the above.
2. The solution to the equation  $\log_2(x) + \log_4(x) = 0$  is:
  - (a) 0.
  - (b)  $\frac{1}{2}$ .
  - (c) 1.
  - (d) 2.
  - (e) Does not exist.
3. When one ounce of water is added to a mixture of acid and water, the new mixture is 20% acid. When one ounce of acid is added to the new mixture, the result is  $33\frac{1}{3}\%$  acid. The percentage of acid in the original mixture is
  - (a) 22%.
  - (b) 24%.
  - (c) 25%.
  - (d) 30%.
  - (e)  $33\frac{1}{3}\%$ .
4. Four suspects, one of whom was known to have committed a crime, made the following statements when questioned by the police. If only one of the suspects is telling the truth, who did it?
  - Doug: Don did it.
  - Darla: I didn't do it.
  - Dora: Don is lying.
  - Don: Dora did it.
  - (a) Doug.
  - (b) Darla.
  - (c) Dora.
  - (d) Don.
  - (e) Nobody.

5. Find  $h$ .

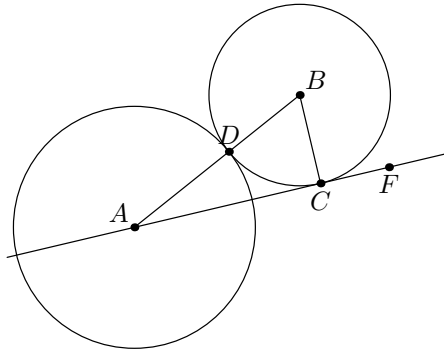


- (a)  $\sqrt{34}/15$ .  
 (b)  $8/15$ .  
 (c)  $8/5$ .  
 (d)  $15/\sqrt{34}$ .  
 (e)  $15/4$ .
6. If  $g\left(\sqrt{\frac{x-1}{x+1}}\right) = 3x$ , what is  $g(3)$ ?
- (a)  $-\frac{15}{4}$ .  
 (b)  $-\frac{5}{4}$ .  
 (c)  $\frac{\sqrt{3}}{3}$ .  
 (d)  $\frac{\sqrt{2}}{2}$ .  
 (e) 9.
7. If one die is fair and the second die is “loaded” so that a “6” is twice as likely as any other number, what is the probability of rolling a “7” with the sum of the dice?
- (a)  $\frac{1}{6}$ .  
 (b)  $\frac{1}{7}$ .  
 (c)  $\frac{4}{21}$ .  
 (d)  $\frac{1}{14}$ .  
 (e)  $\frac{1}{12}$ .
8. How many two-digit numbers  $n$  (between 10 and 99, inclusive) are there such that  $n$  is twice the number gotten by reversing its digits?
- (a) 0.  
 (b) 1.  
 (c) 2.  
 (d) 3.  
 (e) 4.

9. A car has an odometer reading of 15951 miles, which is a palindrome (the number reads the same forward and backward). After two hours of continuous driving at a constant speed, the odometer reaches the next palindrome. How fast was the car being driven during those two hours (in miles per hour rounded to the nearest tenth)?

- (a) 24.5.
- (b) 32.5.
- (c) 40.3.
- (d) 55.
- (e) 60.

10. The graph below has the following characteristics:



- A is the center of a circle with area equal to  $9\pi$  units.
- B is the center of a circle with circumference equal to 16 units.
- D is a point on both circles on the line segment AB.
- The line segment AF is tangent to the circle of center B at the point C.

What is the area of  $\triangle ABC$ ?

- (a) It cannot be determined..
- (b)  $\frac{12}{\pi} \sqrt{1 + \frac{48}{\pi}}$ .
- (c) 6.
- (d)  $\frac{4}{\pi} \sqrt{9 + \frac{48}{\pi}}$ .
- (e) It can be determined, but the answer is none of the above.

11. If  $t = \frac{1}{1 - \sqrt[4]{2}}$ , then  $t$  equals

- (a)  $(1 - \sqrt[4]{2})(2 - \sqrt{2})$ .
- (b)  $(1 - \sqrt[4]{2})(1 + \sqrt{2})$ .
- (c)  $(1 + \sqrt[4]{2})(1 - \sqrt{2})$ .
- (d)  $(1 + \sqrt[4]{2})(1 + \sqrt{2})$ .
- (e)  $-(1 + \sqrt[4]{2})(1 + \sqrt{2})$ .

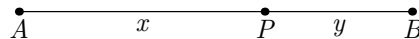
12. What is the remainder when  $2^{2006}$  is divided by 17?

- (a) 0.
- (b) 1.
- (c) 5.
- (d) 11.
- (e) 13.

13. Find the sum of the infinite series

$$\frac{2}{1} + \frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{2}{81} + \frac{1}{243} + \dots = \sum_{n=0}^{\infty} \frac{\frac{3}{2} + \frac{1}{2}(-1)^n}{3^n}$$

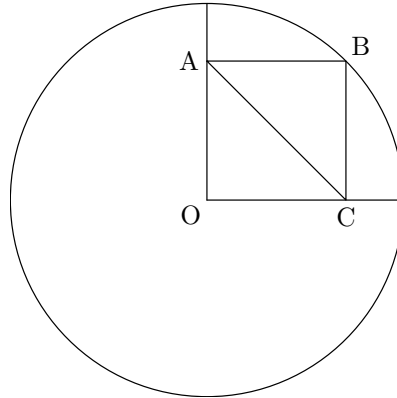
- (a)  $\frac{7}{2}$ .  
 (b)  $\frac{21}{8}$ .  
 (c)  $\frac{17}{4}$ .  
 (d)  $\frac{9}{8}$ .  
 (e)  $\frac{3}{2}$ .
14. If  $g(x) = 1 - x^2$  and  $f(g(x)) = \frac{1-x^2}{x^2}$  when  $x \neq 0$ , then  $f(1/2)$  equals
- (a)  $\frac{\sqrt{2}}{2}$ .  
 (b)  $\frac{3}{4}$ .  
 (c) 1.  
 (d)  $\sqrt{2}$ .  
 (e) 3.
15. A playing card such as the five of hearts has both a rank (five) and a suit (hearts). A full house is a hand of five cards where three cards share the same rank, and the remaining two cards themselves share the same rank, e.g., three aces and two kings.  
 The seven's and eight's are taken from a deck and shuffled; then a five-card hand is dealt from them. What is the probability that the hand is a full house?
- (a)  $\frac{1}{7}$ .  
 (b)  $\frac{1}{6}$ .  
 (c)  $\frac{2}{7}$ .  
 (d)  $\frac{8}{15}$ .  
 (e)  $\frac{6}{7}$ .
16. The line segment  $\overline{AB}$  is divided into two pieces by a point  $P$ . Let  $x$  and  $y$  denote the length of the longer and the shorter pieces as indicated in the picture below (which is *not* drawn to scale). If  $\frac{x+y}{x} = \frac{x}{y}$ , find the ratio  $\frac{y}{x}$ .



- (a)  $\frac{3 - \sqrt{5}}{2}$ .  
 (b)  $\frac{3}{5}$ .  
 (c)  $\frac{\sqrt{5} - 1}{2}$ .  
 (d)  $\frac{2}{3}$ .  
 (e)  $\frac{4}{5}$ .

17. A boy has as many brothers as he has sisters. His sister has twice as many brothers as she has sisters. How many children are in this family?
- (a) 3.
  - (b) 4.
  - (c) 5.
  - (d) 6.
  - (e) 7.
18. The equation  $x^3 = 9 + 46i$ , where  $i = \sqrt{-1}$  has a solution of the form  $a + bi$  where  $a$  and  $b$  are integers. What is  $a + b$ ?
- (a) -1.
  - (b) 0.
  - (c) 3.
  - (d) 5.
  - (e) None of the these.
19. The winning team of the World Series must win four games out of seven. Assuming that the two teams in the World Series are equally matched, find the probability that the Series lasts exactly six games.
- (a)  $\frac{1}{32}$ .
  - (b)  $\frac{5}{32}$ .
  - (c)  $\frac{15}{32}$ .
  - (d)  $\frac{5}{16}$ .
  - (e)  $\frac{15}{64}$ .
20. How many digits are there (in decimal form) in  $2^n$  where  $n = 1,000,000$ ?
- (a)  $1,000,000 (\log_{10}(2))$ .
  - (b)  $1,000,000 (\log_{10}(2)) + 1$ .
  - (c)  $1,000,000 (\log_2(10))$ .
  - (d)  $1,000,000 (\log_2(10)) + 1$ .
  - (e) None of the above.
21. Two comets were discovered orbiting the sun in 2000. The path of one can be represented by the parametric function  $(\sqrt{6} \cos(t), \sqrt{48} \sin(t) - \sqrt{42})$  where  $t$  is in years after the beginning of 2000 and the distance is measures in astronomical units. Using the same units, the second follows a path given by  $x(t) = \sqrt{24} \cos(t) - 3\sqrt{2}$  and  $y(t) = \sqrt{6} \sin(t)$ . Will the two comets ever crash?
- (a) No.
  - (b) Yes, they already did (i.e. between the beginning of 2000 and 2010).
  - (c) Yes, they will before 2050.
  - (d) Yes, they will between 2050 and 2100.
  - (e) Yes, they will after 2100.

22. A rectangular fish tank with length  $x$  inches, width  $y$  inches, and height  $z$  inches holds 10 gallons. (A gallon has 231 cubic inches in it.) A second fish tank has one inch less length and one inch more width, and also holds 10 gallons. In the first tank, how much larger is the width than the length?
- (a) 0.1 inches.  
 (b) 0.231 inches.  
 (c) 1 inch.  
 (d) 2.31 inches.  
 (e) 10 inches.
23. In the figure below, the radius of the circle is 10. What is the length of the diagonal AC of the rectangle OABC?



- (a)  $\sqrt{2}$ .  
 (b)  $\sqrt{10}$ .  
 (c)  $5\sqrt{2}$ .  
 (d) 10.  
 (e)  $10\sqrt{2}$ .
24. The function  $f$  is defined recursively as follows:

$$f(n+1) = \frac{3[f(n)] + 1}{3},$$

for  $n = 1, 2, 3, \dots$  and  $f(1) = 7$ . Find  $f(100)$ .

- (a) 33.  
 (b) 40.  
 (c) 43.  
 (d) 304.  
 (e) 307.
25. A fair coin is tossed 17 times. Determine the probability of obtaining at least 11 consecutive heads.
- (a)  $\frac{1}{128}$ .  
 (b)  $\frac{1}{256}$ .  
 (c)  $\frac{1}{512}$ .  
 (d)  $\frac{1}{1024}$ .  
 (e)  $\frac{1}{2048}$ .

26. Suppose hops, skips, and jumps are specific units of length. Let  $b$  hops equal 1 skip,  $c$  skips equal  $d$  jumps, and  $e$  jumps equal 3 meters. If Molly takes 2 hops, 4 skips, and 1 jump, how far has she traveled in meters?

- (a)  $\frac{2bce + 4ce + de}{3d}$ .  
 (b)  $\frac{6d + 12bd + 3bc}{bce}$ .  
 (c)  $\frac{18d + 3bc}{bce}$ .  
 (d)  $\frac{ec(2b + 4)}{3d}$ .  
 (e) None of the above.

27. Suppose that  $f$  is a function such that  $f(\cos(x)) = \cos(17x)$ . Which of the functions  $g$  has the property that  $g(\sin(x)) = \sin(17x)$ ?

- (a)  $g(x) = f(\sqrt{1-x^2})$ .  
 (b)  $g(x) = f(x - \frac{\pi}{4})$ .  
 (c)  $g(x) = \sqrt{1-f(x)^2}$ .  
 (d)  $g(x) = f(x)$ .  
 (e)  $g(x) = -f(x)$ .

28. Given that  $0.475 < \log_{10} 3 < 0.478$ , how many digits are in the number  $3^{50}$ ?

- (a) 18.  
 (b) 23.  
 (c) 24.  
 (d) 36.  
 (e) 50.

29. In isosceles right triangle  $ABC$ ,  $M$  is the midpoint of hypotenuse  $\overline{AB}$ . An equilateral triangle has one vertex on  $\overline{AC}$ , one on  $\overline{BC}$ , and one at  $M$ . If  $\overline{AB}=24$  and if the length of one side of the equilateral triangle is  $k(\sqrt{3}-1)$ , find  $k$ .

- (a) 9.  
 (b) 10.  
 (c) 11.  
 (d) 12.  
 (e) 13.

$\theta$ (degrees)	$\sin(\theta)$	$\cos(\theta)$
0	0	1
15	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
75	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
90	1	0

30. The median of three test scores is 86, while the range is 12. If  $\bar{x}$  is the mean of the three test scores, then what values can  $\bar{x}$  take?

- (a) [74, 98].  
 (b) [82, 90].  
 (c) [83, 89].  
 (d) [84, 88].  
 (e) [85, 87].

31. If  $4^{3x} = 8^{y+10}$ , what is  $2x - y$ ?
- (a)  $-2$ .
  - (b)  $4$ .
  - (c)  $10$ .
  - (d)  $80$ .
  - (e) Not enough information.
32. Two seven-sided dice each have faces numbered 1 through 7. When the dice are rolled, each face has an equal probability of appearing on the top. What is the probability that the sum of the two top numbers is greater than or equal to their product?
- (a)  $\frac{1}{7}$ .
  - (b)  $\frac{13}{49}$ .
  - (c)  $\frac{2}{7}$ .
  - (d)  $\frac{16}{49}$ .
  - (e)  $\frac{1}{3}$ .
33. For all real nonzero numbers,  $f(x) = 1 - \frac{1}{x}$  and  $g(x) = 1 - x$ . If  $h(x) = f[g(x)]$ , for what value of  $x$  does  $h(x) = 8$ ?
- (a)  $\frac{7}{8}$ .
  - (b)  $\frac{6}{7}$ .
  - (c)  $\frac{10}{9}$ .
  - (d)  $\frac{7}{6}$ .
  - (e)  $\frac{8}{7}$ .
34. A cone set on its base is filled to half its height with water. If the diameter of the base is 8 m and the height of the cone is 4 m, and water weighs 10,000 newtons/m<sup>3</sup>, then what is the weight of the water in the cone?
- (a)  $\frac{64\pi}{3}$  newtons.
  - (b)  $\frac{224\pi}{3}$  newtons.
  - (c)  $\frac{560,000\pi}{3}$  newtons.
  - (d)  $\frac{640,000\pi}{3}$  newtons.
  - (e)  $\frac{2,240,000\pi}{3}$  newtons.
35. At Nat's Nuts, a  $2\frac{1}{4}$  pound bag of pistachio nuts cost \$6.00. At this rate, what is the cost in cents of a bag weighing 9 ounces? Note that there are 16 ounces in a pound.
- (a) 1.5.
  - (b) 24.
  - (c) 150.
  - (d) 1350.
  - (e) 2400.