

Comprehensive Mathematics Contest

Elon University Mathematics and Statistics Department

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Multiple Choice

- Let a and b be lengths of the major and minor axis of the ellipse given by $4x^2 + 9y^2 - 16x + 36y + 16 = 0$. What is the value of ab ?
 - 2.
 - 3.
 - 6.
 - 12.
 - 24.
- For which nonzero real numbers x is $\frac{|x-|x||}{x}$ a positive integer?
 - $x < 0$ only.
 - $x > 0$ only.
 - x a nonzero even integer.
 - all x except zero.
 - no nonzero x .
- An equilateral triangle is circumscribed about a circle of radius 9. What is the area of the triangle?
 - $27\sqrt{3}$.
 - $81\sqrt{3}/4$.
 - $243\sqrt{3}$.
 - $729\sqrt{3}/4$.
 - none of the above.
- The base 7 representation of a positive integer has 2 digits. Reversing those digits gives the base 5 representation the same number. How many such numbers have a base 10 representation less than 2018?
 - 1.
 - 5.
 - 72.
 - 118.
 - 144.
- On the front of a building there are 5 columns that are marked from right to left with the letters A through E, in standard order. Start counting the columns at A until you reach F, then reverse directions and continue to count (only counting F once) until you get back to A. Reverse directions again and continue counting (again, only counting A once). If you continue counting in this manner, which of the lettered columns will be counted number 983?
 - A.
 - B.
 - C.
 - D.
 - E.

6. The diameter of a circle 16 inches. By how much should the diameter be decreased in order to reduce the area by 39π square inches?

- (a) 3.
- (b) 4.
- (c) 5.
- (d) 6.
- (e) none of these.

7. What are the last 2 digits of 2018^{2018} ?

- (a) 24.
- (b) 32.
- (c) 64.
- (d) 72.
- (e) 74.

8. If n is a non-negative integer and

$$(f(n))^2 - f(n-1)f(n+1) = (-2)^n$$

for all n , with $f(0) = f(1) = 1$, then what is the value of $f(3)$?

- (a) -3.
- (b) 1.
- (c) 3.
- (d) 5.
- (e) 13.

9. A water reed sticks out of the water by 1 foot. If the reed does not bend when it is pulled to one side the top reaches the surface 3 feet from the where it originally protruded. How deep (in feet) is the water?

- (a) 3.
- (b) 4.
- (c) 5.
- (d) 7.
- (e) 12.

10. Let a, b be positive integers. Define a binary operation, $*$, by $a * b = 2ab - b$. How many solutions are there to $a * b = 2018$?

- (a) 0.
- (b) 1.
- (c) 2.
- (d) 3.
- (e) 4.

11. Let A and B be two points on a circle of radius 3 with $m\widehat{AB} = 36^\circ$. If a point C on the circle with $C \neq A$ and $C \neq B$ is chosen at random, what is the probability that $m\angle BCA = 18^\circ$?
- (a) 0.8.
 - (b) 0.9.
 - (c) 0.95.
 - (d) 1.
 - (e) None of the above.
12. What is the largest integer n so that 5^n divides into $100!$.
- (a) 21.
 - (b) 22.
 - (c) 23.
 - (d) 24.
 - (e) None of the above.
13. Suppose ABC is an isosceles triangle such that $AC = BC$. Suppose BCD is another isosceles triangle such that $BC = BD$. Suppose also that BD intersects AC at a right angle. If $\angle A = 59^\circ$, what is $\angle D$??
- (a) 55° .
 - (b) 62° .
 - (c) 69° .
 - (d) 76° .
 - (e) 83° .
14. Let $f(x) = x^5 - x^4 + 2x^3 - 3x^2 + 4x - 5$ and let a, b, c, d, e be the complex roots of $f(x)$. What is $a^2 + b^2 + c^2 + d^2 + e^2$?
- (a) -5 .
 - (b) -3 .
 - (c) -1 .
 - (d) 2.
 - (e) 4.
15. How many positive divisors of 2018^{2018} are less than 4,000,000?
- (a) 12.
 - (b) 22.
 - (c) 32.
 - (d) 34.
 - (e) None of the above.

16. Let Triangle A have sides of 5, 7, and 10. Suppose the area of Triangle A is written in the form $a\sqrt{b}$ where b is square free. What is $a+b$?
- (a) 23.
 (b) 33.
 (c) 45.
 (d) 66.
 (e) 94.

17. What is the sum of all positive integers less than 100 that have exactly 12 positive divisors?
- (a) 144.
 (b) 168.
 (c) 252.
 (d) 312.
 (e) none of the above.

18. Let $f(x)$ be the function from real numbers to real numbers defined by:

$$f(x) = \begin{cases} x + 2, & \text{if 3 is a divisor of } [x] \\ x - 1, & \text{otherwise} \end{cases}$$

where $[x]$ denotes the greatest integer less than or equal to x . Let $f^{(n)}$ denote f composed with itself n times. So $f^{(2)}(x) = f(f(x))$, $f^{(3)}(x) = f(f(f(x)))$, and so on. What is $f^{(2018)}(\pi)$?

- (a) $\pi - 1$.
 (b) π .
 (c) $\pi + 1$.
 (d) $\pi + 2$.
 (e) none of the above.
19. Triangle ABC is isosceles with base AC . Suppose there exists a point P on BC and a point Q on AB such that $AC = AP = PQ = QB$. What is $\angle B$?
- (a) $45/2^\circ$.
 (b) 25° .
 (c) $55/2^\circ$.
 (d) 29° .
 (e) None of the above.

20. What is the value of $\sum_{j=0}^{11} \sum_{i=0}^j \binom{j}{i}$?

- (a) 511.
 (b) 1023.
 (c) 2047.
 (d) 4095.
 (e) 8191.

21. Let A_1 denote the area of a regular hexagon inscribed in a circle; let A_2 denote the area of a regular hexagon circumscribed about that same circle. The ratio of A_1 to A_2 is
- $\frac{3}{4}$.
 - $\frac{\sqrt{3}}{4}$.
 - $\frac{\pi}{2}$.
 - $\frac{3}{5}$.
 - $\frac{4}{5}$.
22. If $\frac{A}{2x-3} + \frac{B}{x+4} = \frac{11x+22}{(x+4)(2x-3)}$, then $A - B$ equals
- 9.
 - 5.
 - 5.
 - 9.
 - none of the above.
23. Let $f(x) = \frac{1}{2+x}$. Let $f^{(n)}$ denote f composed with itself n times. So $f^{(2)}(x) = f(f(x))$, $f^{(3)}(x) = f(f(f(x)))$, and so on. What is $\lim_{n \rightarrow \infty} f^{(n)}(2018)$?
- $\frac{1}{2020}$.
 - $\sqrt{2} - 1$.
 - $\sqrt{2} + 1$.
 - $\frac{\sqrt{5} + 2}{2}$.
 - $\sqrt{5} + 1$.
24. A ball of radius r is tangent to the floor and one wall of the room. Find, in terms of r , the radius of the largest sphere that can be rolled through the space between the ball, the wall, and the floor.
- $r/\sqrt{2}$.
 - $r(1 + \sqrt{2})$.
 - $r(2 - \sqrt{2})$.
 - $r(3 - 2\sqrt{2})$.
 - $r(5 - 3\sqrt{2})$.
25. You are walking on the sidewalk of a city street. The probability you will see a car within the next 20 minutes is $609/625$. What is the probability you will see a car within the next 5 minutes? Assume the probability of seeing a car at any moment is the same for the entire 20 minutes.
- $6/625$.
 - $609/2500$.
 - $2/5$.
 - $3/5$.
 - none of the above.

Integer Answers

26. How many solutions are there that satisfy $\sin(4x) = 0$ and $0 < x < 2\pi$?
27. The product of 2 numbers is 2018. When 1 is subtracted from each the product is 1900. What is the sum of the squares of the two numbers?
28. Suppose the integer $a!/b!$ is a multiple of 4 but not a multiple of 8. What is the maximum value of $a - b$?
29. How many different ways can three brothers and 5 other boys be seated in a row with no two brothers seated next to each other?
30. Let S be the set of all positive three-digit integers where each digit is odd. What is the sum of all the members of S ?