

# Comprehensive Mathematics Contest

Elon University Mathematics and Statistics Department

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## Multiple Choice

- If  $131(\text{base } 4)$  is added to  $61(\text{base } 8)$ , then what is the sum written in base 8?
  - 35.
  - 78.
  - 116.
  - 212.
  - none of the above.
- What is the area of the triangle given by the points  $(0,3)$ ,  $(0,7)$  and  $(5,0)$ ?
  - 0.
  - 7.
  - $21/2$ .
  - $\sqrt{1073}$ .
  - $35/2$ .
- The polynomial  $(x + y)^7$  is expanded in decreasing powers of  $x$ . The second and third terms have equal value when evaluated at  $x = p$  and  $y = q$ . Suppose  $p + q = 1$  with  $p$  and  $q$  positive. What is the value of  $p$ ?
  - $1/4$ .
  - $1/3$ .
  - $2/3$ .
  - $3/4$ .
  - $6/7$ .
- What are the last two digits of  $2019^{2019}$ ?
  - 19.
  - 39.
  - 59.
  - 79.
  - 99.
- How many positive integers  $n$  satisfy the following condition:
$$(2019n)^{100} > n^{200} > 4^{300}?$$
  - 2010.
  - 2012.
  - 2016.
  - 2019.
  - 2020.
- A pair of die are tossed. If at least one die is less than 3, what is probability that the sum is 7?
  - 0.
  - $1/9$ .
  - $1/6$ .
  - $1/5$ .
  - None of the above.

7. What is the value of  $\left(\frac{1+i}{-1+i}\right)^{2019}$
- $1 + i$ .
  - $i$ .
  - $-1 + i$ .
  - $-1 - i$ .
  - none of these.
8. What is the value of  $k$  so that the circle  $x^2 + y^2 = 9$  intersects the parabola  $y = x^2 + k$  in exactly three points.
- 9.
  - 3.
  - 0.
  - 3.
  - 9.
9. The negation of the statement, "Some people are good at math", is
- Some people are bad at math.
  - All people are good at math.
  - All people are bad at math.
  - No one is bad at Math.
  - none of the these.
10. When the mean, median, and mode of the list

10,2,5,2,4,2,x

are arranged in increasing order, they form a non-constant arithmetic progression. What is the sum of all possible real values of  $x$ ?

- 3.
  - 6.
  - 9.
  - 17.
  - 20.
11. If both  $x$  and  $y$  are integers, how many solutions are there for the equation  $(x - 8)(x - 10) = 2^y$ ?
- 0.
  - 1.
  - 2.
  - 3.
  - None of the above.
12. A polynomial of degree three with leading coefficient 1 and integer coefficients has one real root, which is also an integer. Which of the following can also be a root of the polynomial?
- $\frac{1+i\sqrt{11}}{2}$ .
  - $\frac{1+i}{2}$ .
  - $\frac{1}{2} + i$ .
  - $1 + \frac{i}{2}$ .
  - $\frac{1+i\sqrt{13}}{2}$ .

13. If  $x$ ,  $2x + 3$  and  $3x + 6$  are in geometric progression, then what is the fifth term?
- (a) -3.
  - (b) -1.
  - (c) 0.
  - (d) 3.
  - (e) 6.
14. The length of the sides of a triangle are 10, 24 and 26. What is the area of the triangle?
- (a) 60.
  - (b) 120.
  - (c) 240.
  - (d) 360.
  - (e) none of the above.
15. How many positive integers less than ten million have exactly 55 positive divisors?
- (a) 0.
  - (b) 1.
  - (c) 2.
  - (d) 3.
  - (e) 4.
16. If  $a = \log_8(225)$  and  $b = \log_2(15)$ , then  $a$  is equal to
- (a)  $b/2$ .
  - (b)  $2a/3$ .
  - (c)  $2b/3$ .
  - (d)  $b/4$ .
  - (e) None of the above.
17. Let  $A, B, C, D$  be four coplanar points, no three of which are collinear. The point  $D$  is in the interior of triangle  $ABC$ . Of the four possible triangles, what is the minimum number of obtuse triangles?
- (a) 0.
  - (b) 1.
  - (c) 2.
  - (d) 3.
  - (e) 4.
18. The polynomial  $p(x) = x^4 + ax^3 + 4x^2 + bx + c$  has the property that the average of its zeros, the product of its zeros, and the sum of its coefficients are all equal. The  $y$ -intercept of the graph of  $y = p(x)$  is 2. What is  $b$ ?
- (a) 1.
  - (b) 3.
  - (c) 5.
  - (d) 7.
  - (e) None of the above.

19. How many digits are in the product of 3,161,694,620,827,086,886 and 316,286,069,950,967?

- (a) 32.
- (b) 33.
- (c) 34.
- (d) 35.
- (e) None of the above.

20. Let  $f(x) = x^7 + 3x^6 + 3x^4 + 3$  and let  $a_1, \dots, a_7$  be the (possibly complex) roots of  $f(x)$ . What is  $a_1^3 + \dots + a_7^3$ ?

- (a) -36.
- (b) -27.
- (c) -9.
- (d) 0.
- (e) None of the above.

21. The ratio of the interior angles of two regular polygons is 4 : 3. How many such pairs are there?

- (a) 1.
- (b) 2.
- (c) 3.
- (d) 4.
- (e) None of the above.

22. Let  $f(x) = \frac{cx}{8x - 16}$  for  $x \neq 2$  and  $c$  is a constant. If  $f(f(x^2)) = x^2$  for all  $x > 2$ , then what is the value of  $c$ ?

- (a) -16.
- (b) -8.
- (c) 2.
- (d) 8.
- (e) 16.

23. How many triples  $(a, b, c)$  of positive integers satisfy the simultaneous equations

$$ab + bc = 44$$

$$ac + bc = 23$$

- (a) 0.
- (b) 1.
- (c) 2.
- (d) 3.
- (e) None of the above.

24. Square  $ABCD$  has unit length sides. Point  $E$  on side  $AB$  and point  $F$  on side  $AD$  are chosen so that  $AE = AF$  and quadrilateral  $CDFE$  has maximum area. What is this maximum area?

- (a)  $9/16$ .
- (b)  $5/8$ .
- (c)  $2/3$ .
- (d)  $3/4$ .
- (e) None of the above.

25. Suppose  $0 < a < 1$ . Let  $b = a^a$ ,  $c = a^b$ , and  $d = a^c$ . Arrange  $a$ ,  $b$ ,  $c$ , and  $d$  in order of increasing magnitude.

- (a)  $a, b, c, d$ .
- (b)  $d, c, b, a$ .
- (c)  $a, c, d, b$ .
- (d)  $a, d, c, b$ .
- (e) None of the above.

## Integer Answers

26. Let  $S$  be the solution set for  $\log_6(P) + \log_6(P + 5) = 2$ . What is the product of the elements of  $S$ .
27. A line with slope 6 bisects the area of the unit square with vertices  $(0,0)$ ;  $(0,1)$ ;  $(1,1)$ ;  $(1,0)$ . The  $y$ -intercept of this line is a rational number  $a/b$  with  $b > 0$ , expressed in lowest terms. What is  $b$ ?
28. Six students in class have name tags. The teacher asks each student to take off their name tag and place it into a hat. The six students then take one of the name tags. If the probability of exactly one person getting their own name tag is given in lowest terms by  $\frac{p}{q}$ , what is  $p + q$ ?
29. Let  $\lfloor x \rfloor$  be the integer that is less than or equal to  $x$ . What is the value of  $\sum_{n=1}^{2019} \lfloor \log_{10}(n) \rfloor$ ?
30. Each face of a cube is to be painted a different solid color. How many different looking cubes can be created with 6 different colors?